Math 246C Lecture 13 Notes

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1 The Uniformization Theorem

1.1 Uniformization, Case 1

Let's finish the proof of the first case of the Uniformization theorem.

Theorem 1.1 (Uniformization, Case 1)**.** *Let X be a simply connected Riemann surface. The following conditions are equivalent:*

1. *$G_x(y)$ exists for some $x \in X$.*
2. *$G_x(y)$ exists for all $x \in X$.*
3. *There exists a holomorphic bijection $\varphi : X \to \{z : |z| < 1\}$.*

Proof. (1) $\implies$ (3): Let $D \subseteq X$ be a parametric disc with $x \in D$ and $z(x) = 0$. We saw last time that there is a $\varphi_D \in \mathrm{Hol}(D)$ such that $|\varphi_D(y)| = e^{-G_x(y)}$ for all $y \in D$. If $D' \subseteq X$ is a parametric disc such that $x \notin D$, then there exists $\varphi_{D'} \in \mathrm{Hol}(D')$ such that $|\varphi_{D'}(y)| = e^{-G_x(y)}$ for all $y \in D'$: $G|_{D'}$ is harmonic, so $G_x = \mathrm{Re}(f_{D'})$ with $f_{D'}$ holomorphic, and we can take $\varphi_{D'}(y) = e^{-f_{D'}(y)}$. On $D \cap D'$, $\varphi_D/\varphi_{D'}$ is holomorphic with modulus 1. So $\varphi_D/\varphi_{D'} = e^{i\theta}$ for some θ.

Let γ be a path in X with $\gamma(0) = x$. Then, by compactness, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and parametric discs D_j, $1 \leq j \leq n$, such that $\gamma([t_{j-1}, t_j]) \subseteq D_j$. It follows that φ_D can be continued analytically along all paths in X starting at x. By the monodromy theorem, there is a globally defined holomorphic function $\varphi \in \mathrm{Hol}(X)$ such that $|\varphi(y)| = e^{-G_x(y)}$ for all $y \in X$.

We claim that φ is injective. We have that $\varphi(x) = 0$, and if $\varphi(y) = \varphi(y) = \varphi(x) = 0$, then $y = x$ (since G_x is only infinite at x). Let $z \in X$ with $z \neq x$. Then $|\varphi(z)| < 1$. Consider

$$\varphi_1(y) = \frac{\varphi(y) - \varphi(z)}{1 - \overline{\varphi(z)}\varphi(y)}.$$

Then $\varphi_1 \in \mathrm{Hol}(X)$, and $|\varphi_1| < 1$. Take $v \in \mathcal{F}_z$, the Perron family used to construct G_z. The function $v(y) + \log|\varphi_1(y)|$ is subharmonic on $X \setminus \{z\}$, bounded above, and ≤ 0 far away. By the Lindelöf maximum principle, $v(y) + \log|\varphi_1(y)| \leq 0$ on $X \setminus \{z\}$. So G_z exists, and $G_z(y) + \log|\varphi_1(y)| < 0$. For $y = x$, we get

$$G_z(x) \leq -\log|\varphi_1(x)| = \log|\varphi(z)| = G_x(z).$$

Switching the roles of x and z, we get[1]

$$G_z(x) = G_x(z).$$

The function $G_z(y) + \log|\varphi_1(y)| \leq 0$ is subharmonic for $y \neq z$, and when $y = x$, we have

$$G_z(x) + \log|\varphi_1(x)| = G_x(z) + \log|\varphi(z)| = 0.$$

By the maximum principle, we get

$$G_z(y) = -\log|varphi_1(y)|, \qquad y \neq z.$$

If $\varphi(w) = \varphi(z)$, then $\varphi_1(w) = 0$. So $G_z(w) = \infty$, which means $w = z$.

We have that $\varphi : X \to D = \{|z| < 1\}$ is holomorphic and injective. We do not actually need to prove surjectivity because of the following trick.[2] $\varphi(X) \subseteq D$ is open and simply connected. By the Riemann mapping theorem, there is a holomorphic bijection $\psi : \varphi(X) \to D$. So the map $\psi \circ \varphi \in \mathrm{Hol}(X)$ works. □

Remark 1.1. This is sometimes called the hyperbolic case since D admits a hyperbolic metric. So we have shown that every simply connected manifold that carries a Green's function is conformally equivalent to a space with a hyperbolic metric.

1.2 Uniformization, Case 2

Theorem 1.2 (Uniformization, Case 2)**.** *Let X be a simply connected Riemann surface for which Green's function does not exist. If X is compact, then there is a holomorphic bijection $X \to \hat{\mathbb{C}}$. If X is not compact, there is a holomorphic bijection $X \to \mathbb{C}$.*

The main idea in the proof is to show the existence of a **dipole Green's function**.

Example 1.1. Consider $\log 1/|z|$ on the Riemann sphere. This has singularities of opposite signs at 0 and ∞.

Lemma 1.1 (existence of a dipole Green's function)**.** *Let X be a Riemann surface, let $x_1, x_2 \in X$ be distinct, and let $z_j : D_j \to \{|z| < 1\}$ for $j = 1, 2$ be parametric discs such that $z_j(x_j) = 0$, snd $\overline{D}_1 \cap \overline{D}_2 = \varnothing$. Then there is a functio $nG_{x_1,x_2}(y)$ which is harmonic on $X \setminus \{x_1, x_2\}$ such that $G_{x_1,x_2}(y) + \log|z_1(y)|$ is harmonic in D_1 and $G_{x_1,x_2}(y) - \log|z_2(y)|$ is harmonic in D_2. Furthermore,*

$$\sup_{y \in X \setminus (D_1 \cup D_2)} G_{x_1,x_2}(y) < \infty.$$

[1] This symmetry of the Green's function is actually true in general, but we will not visit that fact now.
[2] The map is actually surjective, but it would take some more work to prove.

Assuming this lemma, which we will prove later, we can finish the proof of the Uniformization theorem.

Proof. Let G_{x_1,x_2} a dipole Green's function for $x_1 \neq x_2 \in X$. Arguing as in the proof of Case 1, we see that there is a $\varphi \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ (i.e. a meromorphic function on X) such that

$$|\varphi(y)| = e^{-G_{x_1,x_2}(y)}, \qquad y \in X.$$

Then φ has a unique zero at x_1 at x_1 and a unique simple pole at z_2.

Assume that $\varphi : X \to \mathbb{C}$ is injective. Then consider $\varphi(X) \subseteq \hat{\mathbb{C}}$, which is simply connected. If $\hat{\mathbb{C}} \setminus \varphi(X)$ contains more than a single point, composing with a Möbius transformation which sends the point to ∞, we get an injective, holomorphic map from X to a subset of $\mathbb{C}$. By the Riemann mapping theorem, we get a holomorphic bijection to D; however, we assumed no Green's function exists, so we have a contradiction. So we must either have $\varphi(X) = \mathbb{C}$ (after composing with a Möbius transformation) or $\varphi(X) = \hat{\mathbb{C}}$. □

Next time, we will show that φ is injective, to complete the proof.